# Convergence of mimetic finite difference discretizations of the diffusion equation

**Markus Berndt** **Konstantin Lipnikov** **David Moulton**

**Mikhail Shashkov**

Los Alamos National Laboratory
T-7, MS B284, Los Alamos, NM 87545
berndt@lanl.gov, lipnikov@lanl.gov, moulton@lanl.gov, shashkov@lanl.gov

# Contents

# Model diffusion problem

We consider the elliptic equation

$$-\mathrm{div}(\boldsymbol{K}\,\nabla p) = b \qquad \text{in} \qquad \Omega$$

subject to the homogeneous Dirichlet b.c.

$$p = 0 \qquad \text{on} \qquad \partial\Omega.$$

The problem can be reformulated as a system of first order equations:

$$\begin{aligned} \mathrm{div}\,\boldsymbol{f} &= b, \\ \boldsymbol{f} &= -\boldsymbol{K}\,\nabla p. \end{aligned}$$

For simplicity we assume that $\boldsymbol{K} = \boldsymbol{I}$.

# Support operator method

Consider the mathematical identity:

$$\int_\Omega \mathrm{grad}\, p\, \boldsymbol{f}\, \mathrm{d}x = -\int_\Omega \mathrm{div}\, \boldsymbol{f}\, p\, \mathrm{d}x \qquad \forall \boldsymbol{f} \in H_{div}(\Omega),\; p \in H_0^1(\Omega).$$

Global support-operators (SO) methodology (for div & grad):

1. define degrees of freedom for variables $p$ and $\boldsymbol{f}$;
2. equip the discrete spaces for $p$ and $\boldsymbol{f}$ with scalar products $[\cdot\,,\,\cdot]_Q$ and $[\cdot\,,\,\cdot]_X$, respectively;
3. choose a discrete approximation to the divergence operator, the *prime* operator $\mathrm{DIV}: X_d \to Q_d$;
4. derive the discrete approximation of the gradient operator, the *derived* operator $\mathrm{GRAD}: Q_d \to X_d$, from the discrete Green formula:

$$[f^d, \mathrm{GRAD}\, p^d]_X = -[\mathrm{DIV}\, f^d, p^d]_Q \qquad \forall p^d \in Q_d,\; \forall f^d \in X_d.$$

# Support operator method

Consider the mathematical identity:

$$\int_\Omega \operatorname{grad} p\, \boldsymbol{f}\, \mathrm{d}x = -\int_\Omega \operatorname{div} \boldsymbol{f}\, p\, \mathrm{d}x \qquad \forall \boldsymbol{f} \in H_{div}(\Omega),\ p \in H_0^1(\Omega).$$

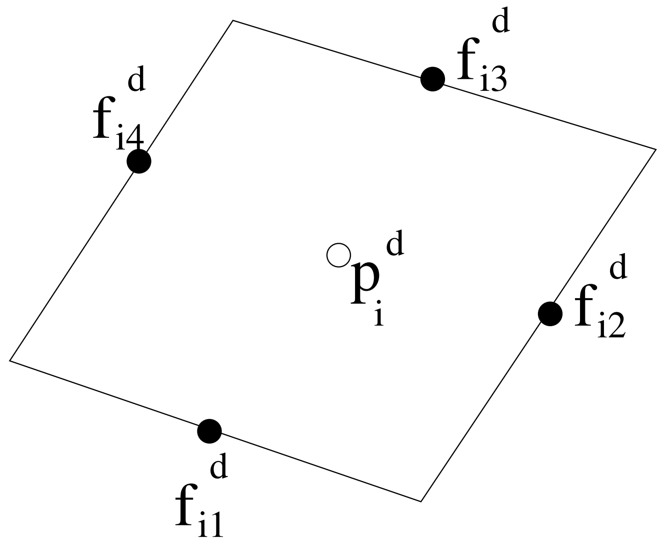
Global support-operators (SO) methodology (for div & grad):

1. define degrees of freedom for variables $p$ and $\boldsymbol{f}$;
2. equip the discrete spaces for $p$ and $\boldsymbol{f}$ with scalar products $[\cdot\,,\ \cdot]_Q$ and $[\cdot\,,\ \cdot]_X$, respectively;
3. choose a discrete approximation to the divergence operator, the *prime* operator $\mathrm{DIV}: X_d \to Q_d$;
4. derive the discrete approximation of the gradient operator, the *derived* operator $\mathrm{GRAD}: Q_d \to X_d$, from the discrete Green formula:

$$[f^d,\ \mathrm{GRAD}\, p^d]_X = -[\mathrm{DIV}\, f^d,\ p^d]_Q \qquad \forall p^d \in Q_d,\ \forall f^d \in X_d.$$

# Mimetic discretizations (1/6)

Step 1 (degrees of freedom for $p$ and $\boldsymbol{f}$).




- $p_i^d$ is defined at a center of cell $e_i$.
- $f_{i1}^d, \ldots, f_{i4}^d$ are defined at mid-points of cell edges. They approximate the normal components of $\boldsymbol{f}$, e.g.

$$f_{i1}^d \approx \boldsymbol{f} \cdot \boldsymbol{n}_{i1}.$$

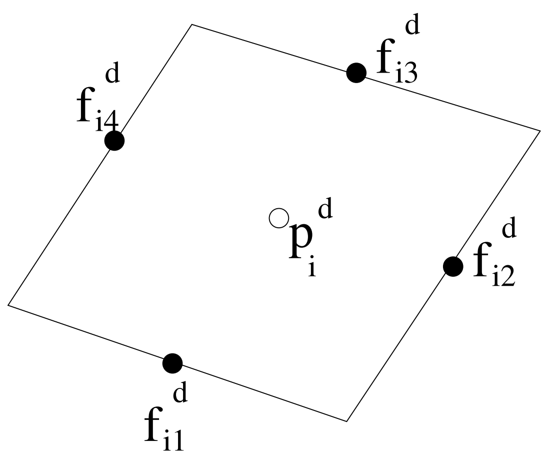
# Mimetic discretizations (2/6)

Step 2 (scalar products for $p^d$ and $f^d$).

- Let $Q_d$ be a vector space of discrete intensities with the scalar product

$$[p^d,\, q^d]_Q = \sum_{i=1}^{N} |e_i|\, p_i^d\, q_i^d \approx \int_\Omega p(x)\, q(x)\, \mathrm{d}x.$$

- Let $X_d$ be a vector space of discrete fluxes with a scalar product $[f^d,\, g^d]_X \approx \int_\Omega \boldsymbol{f}(x) \cdot \boldsymbol{g}(x)\, \mathrm{d}x.$




The vectors can be recovered uniquely at four vertices of quadrilateral $e_i$. Let

$$[f_i^d,\, g_i^d]_{X_{e_i}} = \frac{1}{2} \sum_{j=1}^{4} |T_{ij}|\, \boldsymbol{f}_{ij}^d \cdot \boldsymbol{g}_{ij}^d$$

Then $\quad [f^d,\, g^d]_X = \sum_{i=1}^{N} [f_i^d,\, g_i^d]_{X_{e_i}}.$

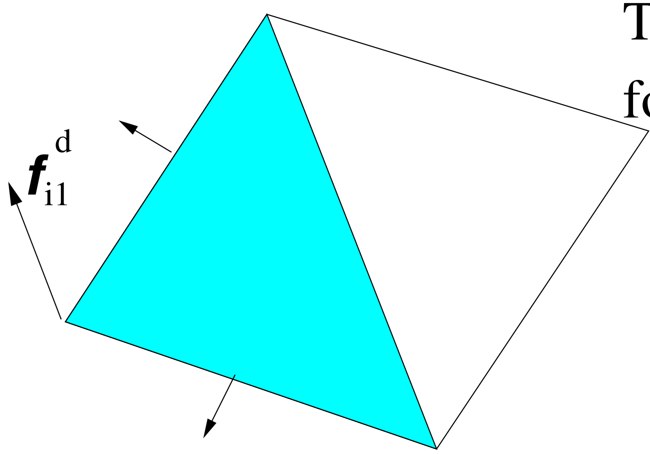
# Mimetic discretizations (2/6)

Step 2 (scalar products for $p^d$ and $f^d$).

- Let $Q_d$ be a vector space of discrete intensities with the scalar product

$$[p^d,\, q^d]_Q = \sum_{i=1}^{N} |e_i|\, p_i^d\, q_i^d \approx \int_\Omega p(x)\, q(x)\, \mathrm{d}x.$$

- Let $X_d$ be a vector space of discrete fluxes with a scalar product $[f^d,\, g^d]_X \approx \int_\Omega \boldsymbol{f}(x) \cdot \boldsymbol{g}(x)\, \mathrm{d}x$.




The vectors can be recovered uniquely at four vertices of quadrilateral $e_i$. Let

$$[f_i^d,\, g_i^d]_{X_{e_i}} = \frac{1}{2} \sum_{j=1}^{4} |T_{ij}|\, \boldsymbol{f}_{ij}^d \cdot \boldsymbol{g}_{ij}^d$$

Then $\quad [f^d,\, g^d]_X = \sum_{i=1}^{N} [f_i^d,\, g_i^d]_{X_{e_i}}.$

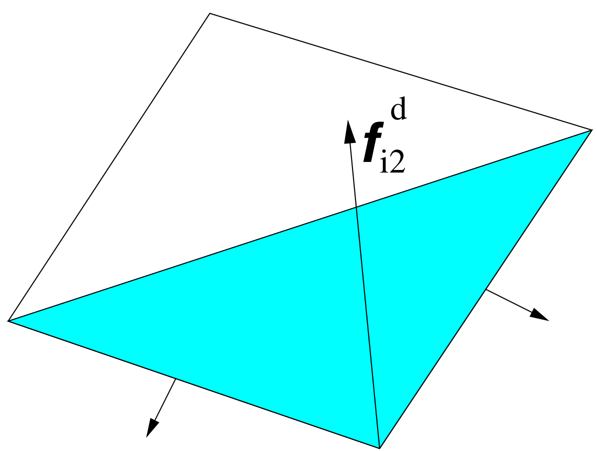
# Mimetic discretizations (2/6)

Step 2 (scalar products for $p^d$ and $f^d$).

- Let $Q_d$ be a vector space of discrete intensities with the scalar product

$$[p^d,\, q^d]_Q = \sum_{i=1}^{N} |e_i|\, p_i^d\, q_i^d \approx \int_\Omega p(x)\, q(x)\, \mathrm{d}x.$$

- Let $X_d$ be a vector space of discrete fluxes with a scalar product $[f^d,\, g^d]_X \approx \int_\Omega \boldsymbol{f}(x)\cdot \boldsymbol{g}(x)\,\mathrm{d}x$.




The vectors can be recovered uniquely at four vertices of quadrilateral $e_i$. Let

$$[f_i^d,\, g_i^d]_{X_{e_i}} = \frac{1}{2}\sum_{j=1}^{4} |T_{ij}|\, \boldsymbol{f}_{ij}^d \cdot \boldsymbol{g}_{ij}^d$$

Then $\quad [f^d,\, g^d]_X = \sum_{i=1}^{N} [f_i^d,\, g_i^d]_{X_{e_i}}.$

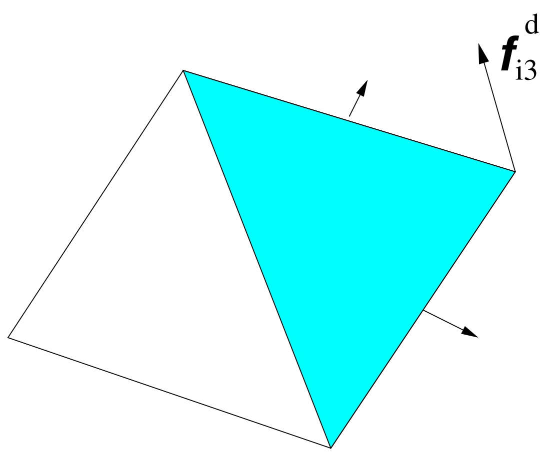
# Mimetic discretizations (2/6)

Step 2 (scalar products for $p^d$ and $f^d$).

- Let $Q_d$ be a vector space of discrete intensities with the scalar product

$$[p^d,\, q^d]_Q = \sum_{i=1}^{N} |e_i|\, p_i^d\, q_i^d \approx \int_\Omega p(x)\, q(x)\, \mathrm{d}x.$$

- Let $X_d$ be a vector space of discrete fluxes with a scalar product $[f^d,\, g^d]_X \approx \int_\Omega \boldsymbol{f}(x) \cdot \boldsymbol{g}(x)\, \mathrm{d}x$.




The vectors can be recovered uniquely at four vertices of quadrilateral $e_i$. Let

$$[f_i^d,\, g_i^d]_{X_{e_i}} = \frac{1}{2} \sum_{j=1}^{4} |T_{ij}|\, \boldsymbol{f}_{ij}^d \cdot \boldsymbol{g}_{ij}^d$$

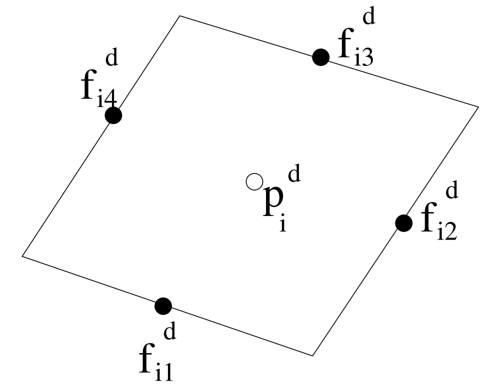
Then $\quad [f^d,\, g^d]_X = \sum_{i=1}^{N} [f_i^d,\, g_i^d]_{X_{e_i}}.$

# Mimetic discretizations (3/6)

Steps 3 & 4 (prime and derived operators).

The prime operator DIV follows from the Gauss theorem:

$$\operatorname{div} \boldsymbol{f} = \lim_{|e| \to 0} \frac{1}{|e|} \oint_{\partial e} \boldsymbol{f} \cdot \boldsymbol{n} \, dl.$$




Center-point quadrature gives

$$\left(\mathrm{DIV}\, f^d\right)_i = \frac{1}{|e_i|} \left( f_{i1}^d \, |l_1| + f_{i2}^d \, |l_2| + f_{i3}^d \, |l_3| + f_{i4}^d \, |l_4| \right)$$

The derived operator GRAD is implicitly given by

$$[f^d, \, \mathrm{GRAD}\, p^d]_X = -[\mathrm{DIV}\, f^d, \, p^d]_Q \qquad \forall p^d \in Q_d, \; f^d \in X_d.$$

# Mimetic discretizations (4/6)

Short summary.

The stationary diffusion problem

$$\begin{aligned} -\operatorname{div} \boldsymbol{K}\nabla p &= b \quad \text{in} \quad \Omega \\ p &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

is rewritten as the 1st order system

$$\boldsymbol{f} = -\boldsymbol{K}\nabla p, \qquad \operatorname{div} \boldsymbol{f} = b$$

and discretized as follows:

$$f^d = -\mathrm{GRAD}\, p^d, \qquad \mathrm{DIV}\, f^d = b^d.$$

# Mimetic discretizations (5/6)

By the definition,

$$[f^d,\, \mathrm{GRAD}\, p^d]_X = -[\mathrm{DIV}\, f^d,\, p^d]_Q.$$

Let $< \cdot\,,\, \cdot >$ be the usual vector dot product. Then

$$[p^d,\, q^d]_Q =< \mathcal{D} p^d,\, q^d >, \qquad [f^d,\, g^d]_X =< \mathcal{M} f^d,\, g^d > .$$

Combining the last two formulas, we get

$$\begin{aligned} [f^d,\, \mathrm{GRAD}\, p^d]_X &= \; < \mathcal{M}\, f^d,\, \mathrm{GRAD}\, p^d > \\ &= \; -[\mathrm{DIV}\, f^d,\, p^d]_Q = - < f^d,\, \mathrm{DIV}^t\, \mathcal{D}\, p^d > . \end{aligned}$$

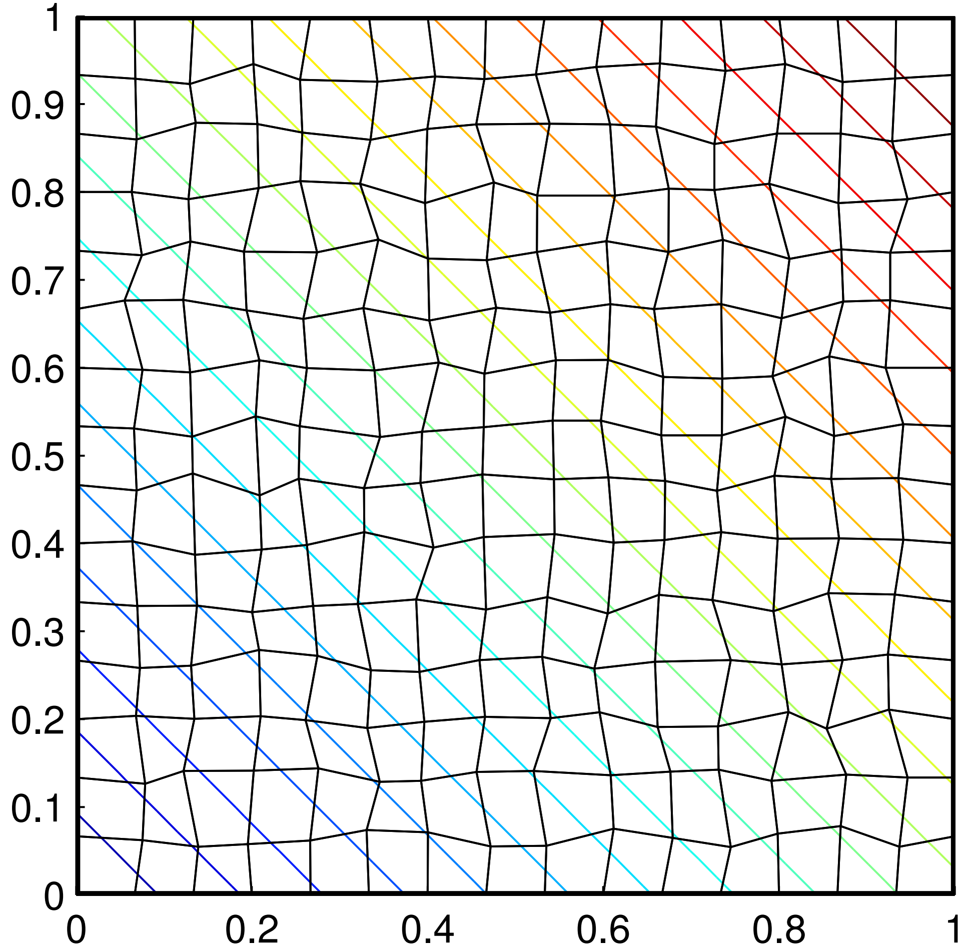
Therefore,

$$\mathrm{GRAD} = -\mathcal{M}^{-1}\, \mathrm{DIV}^t\, \mathcal{D}.$$

is the non-local operator.

# Mimetic discretizations (6/6)

The derived mimetic discretizations are exact for linear solutions.



# Convergence analysis (1/9)

The convergence analysis is based on a connection of the SO method with a mixed finite element (MFE) method:

- the theory of MFE methods justifies the convergence and stability of mimetic discretizations;
- the analysis can not be extended to quadrilateral meshes with non-convex cells and to general polygonal meshes.

# Convergence analysis (2/9)

The system of mimetic finite difference equations

$$f^d = -\mathrm{GRAD}\, p^d, \qquad \mathrm{DIV}\, f^d = b^d$$

is equivalent to the following problem: Find $(f^d, p^d) \in X_d \times Q_d$ such that

$$\begin{aligned} [f^d, g^d]_X + [\mathrm{GRAD} p^d, g^d]_X &= 0, \\ [\mathrm{DIV}\, f^d, q^d]_Q &= [b^d, q^d]_Q, \qquad \forall (g^d, q^d) \in X_d \times Q_d. \end{aligned}$$

Recall that by the definition,

$$[f^d, \mathrm{GRAD}\, p^d]_X = -[\mathrm{DIV}\, f^d, p^d]_Q.$$

# Convergence analysis (3/9)

Thus, the mimetic discretizations result in

$$\begin{aligned}
[f^d,\, g^d]_X - [\mathrm{DIV}\, g^d,\, p^d]_Q &= 0, \\
-[\mathrm{DIV}\, f^d,\, q^d]_Q &= -[b^d,\, q^d]_Q, \qquad \forall (g^d,\, q^d) \in X_d \times Q_d.
\end{aligned}$$

The MFE method with the modified *Raviart-Thomas* finite elements gives

$$\begin{aligned}
(f^h,\, g^h) - (\mathrm{div}\, g^h,\, p^h) &= 0, \\
-(\mathrm{div}\, f^h,\, q^h) &= -(b,\, q^h) \qquad \forall (g^h,\, q^h) \in X_h \times Q_h.
\end{aligned}$$

| | | | | |
|---|---|---|---|---|
| | $p^d$: | at cell centers | $p^h$: | one per cell |
| Degrees of freedom: | $f^d$: | normal components at edge mid-points | $f^h$: | normal components, one per edge |

# Convergence analysis (4/9)

Let $\mathcal{I} : X_d \times Q_d \to X_h \times Q_h$ be the natural isomorphism between the discrete spaces and

$$(g^h, q^h) = \mathcal{I}((g^d, q^d)).$$

Then

$$[\text{DIV}\, g^d, p^d]_Q = (\operatorname{div} g^h, p^h)$$

and

$$[b^d, q^d]_Q = (b, q^h)$$

if $b_i^d$ is the mean value of the source term over the $i$-th mesh cell.

# Convergence analysis (5/9)

Thus, the SO problem: Find $(f^d, p^d) \in X_d \times Q_d$ such that

$$\begin{aligned} [f^d, g^d]_X - [\mathrm{DIV}\, g^d, p^d]_Q &= 0, \\ -[\mathrm{DIV}\, f^d, q^d]_Q &= -[b^d, q^d]_Q, \qquad \forall (g^d, q^d) \in X_d \times Q_d, \end{aligned}$$

can be rewritten as a FE problem: Find $(f^h, p^h) \in X_h \times Q_h$ such that

$$\begin{aligned} (f^h, g^h)_h - (\mathrm{div}\, g^h, p^h) &= 0, \\ -(\mathrm{div}\, f^h, q^h) &= -(b, q^h) \qquad \forall (g^h, q^h) \in X_h \times Q_h, \end{aligned}$$

where

$$(f^h, g^h)_h \equiv [f^d, g^d]_X.$$

# Convergence analysis (6/9)

The FE problem has a unique solution if the following conditions hold:

- continuity:

$$(g^h, g^h)_h \le c_2 (g^h, g^h) \qquad \forall g^h \in X_h;$$

- ellipticity:

$$c_1 (g^h, g^h) \le (g^h, g^h)_h \qquad \forall g^h \in X_h, \quad \mathrm{div}\, g^h = 0;$$

- stability (LBB condition):

$$\sup_{g^h \in X_h} \frac{(\mathrm{div}\ g^h, q^h)}{\|g^h\|_{\mathrm{div}}} \ge c_3 \|q^h\| \qquad \forall q^h \in Q_h.$$

The constants $c_1$, $c_2$ and $c_3$ are independent of $h$.

# Convergence analysis (7/9)

Theorem (Strang). Suppose $\mathcal{T}_h$ is a shape regular triangulation of $\Omega$ and input data are sufficiently smooth. Then

$$\|\boldsymbol{f} - f^h\|_{\text{div}} \leq c \left\{ \inf_{g^h \in X_h} \left[ \|\boldsymbol{f} - g^h\|_{\text{div}} + \Delta(g^h) \right] \right\}$$

and

$$\|p - p^h\| \leq c \left\{ \inf_{q^h \in Q_h} \|p - q^h\| + \inf_{g^h \in X_h} \left[ \|\boldsymbol{f} - g^h\| + \Delta(g^h) \right] \right\}$$

where

$$\Delta(g^h) = \sup_{w^h \in X_h} \frac{|(g^h, w^h) - (g^h, w^h)_h|}{\|w^h\|_{\text{div}}}$$

is the consistency term and $c$ is a positive constant independent of $h$.

# Convergence analysis (8/9)

Lemma. Suppose $\mathcal{T}_h$ is a shape regular quasi-uniform quadrilateral partition of $\Omega$. Then

$$|(g^h, w^h) - (g^h, w^h)_h| \leq ch \, \|g^h\|_1 \, \|w^h\|_{\mathrm{div}},$$

where $c$ is a positive constant independent of $h$.

Thus, the consistency term is small, i.e.

$$\Delta(g^h) \leq c\, h \|g^h\|_1.$$

Remark: For many problems this estimate is very rough.

# Convergence analysis (9/9)

The approximation theory and above lemma result in optimal convergence estimates.

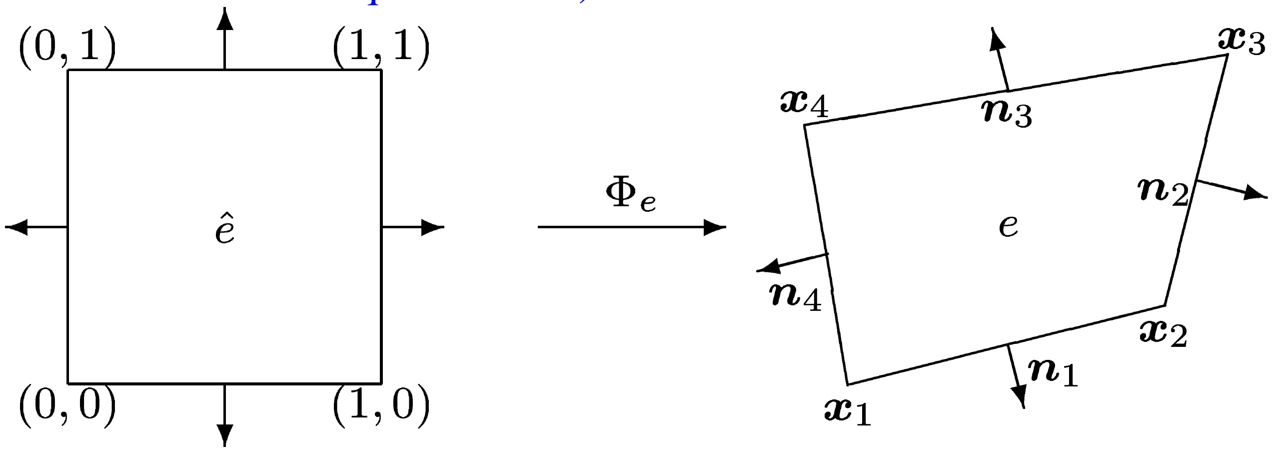
Theorem. Suppose $\mathcal{T}_h$ is a shape regular quasi-uniform quadrilateral partition of $\Omega$ and input data are sufficiently smooth. If $(f^h, p^h) = \mathcal{I}((f^d, p^d))$, then

$$
\begin{aligned}
\|\boldsymbol{f} - f^h\| &\leq c\,h\,\|\boldsymbol{f}\|_1, \\
\|\boldsymbol{f} - f^h\|_{\mathrm{div}} &\leq c\,h\,\{\|\boldsymbol{f}\|_1 + \|\mathrm{div}\,\boldsymbol{f}\|_1\}\,, \\
\|p - p^h\| &\leq c\,h\,\{\|p\|_1 + \|\boldsymbol{f}\|_1\}
\end{aligned}
$$

where $c$ is a positive constant independent of $h$.

# Numerical experiments (1/4)

(Raviart-Thomas elements for a quadrilateral).




$$\Phi_e(\xi,\eta) = \boldsymbol{x}_1(1-\xi)(1-\eta) + \boldsymbol{x}_2\xi(1-\eta) + \boldsymbol{x}_3\xi\eta + \boldsymbol{x}_4(1-\xi)\eta.$$

The Raviart-Thomas finite elements on $\hat{e}$ are

$$\hat{f}_1 = \begin{bmatrix} 0 \\ \eta - 1 \end{bmatrix}, \quad \hat{f}_2 = \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \quad \hat{f}_3 = \begin{bmatrix} 0 \\ \eta \end{bmatrix}, \quad \hat{f}_4 = \begin{bmatrix} \xi - 1 \\ 0 \end{bmatrix}.$$

The Piola transformation is defined by

$$f^h_{e,i} = \frac{|l_i|}{J_e} D\Phi_e\, \hat{f}_i, \qquad i = 1,2,3,4.$$

# Numerical experiments (2/4)

Let the exact solution be

$$p(x,y) = \left(|x-0.5|^{\alpha} - 0.5^{\alpha}\right)\left(|y-0.5|^{\alpha} - 0.5^{\alpha}\right), \qquad 0 \le x, y \le 1,$$

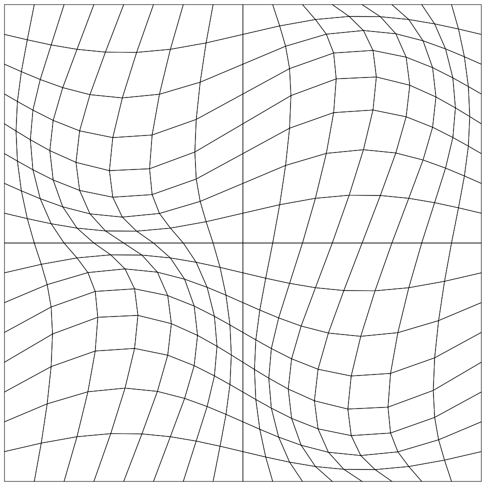
where $\alpha = 2.6$. It is easy to check that

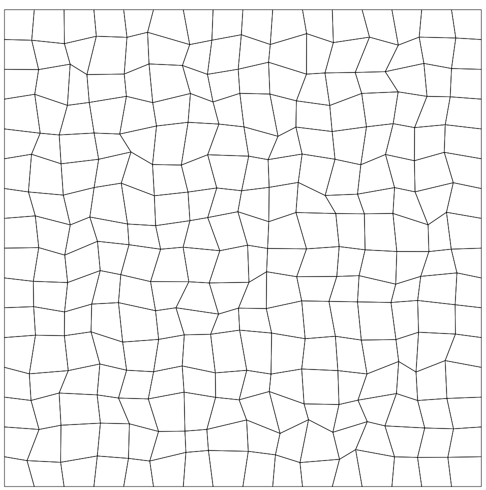
$$\operatorname{div} \boldsymbol{f} = \Delta p \in H^1(\Omega) \qquad \text{and} \qquad \boldsymbol{f} \in (H^1(\Omega))^2.$$

Denote the errors we compute in our experiments by

$$\varepsilon_p = \|p - p^h\| \qquad \text{and} \qquad \varepsilon_{\boldsymbol{f}} = \|\boldsymbol{f} - f^h\|_{\mathrm{div}}.$$

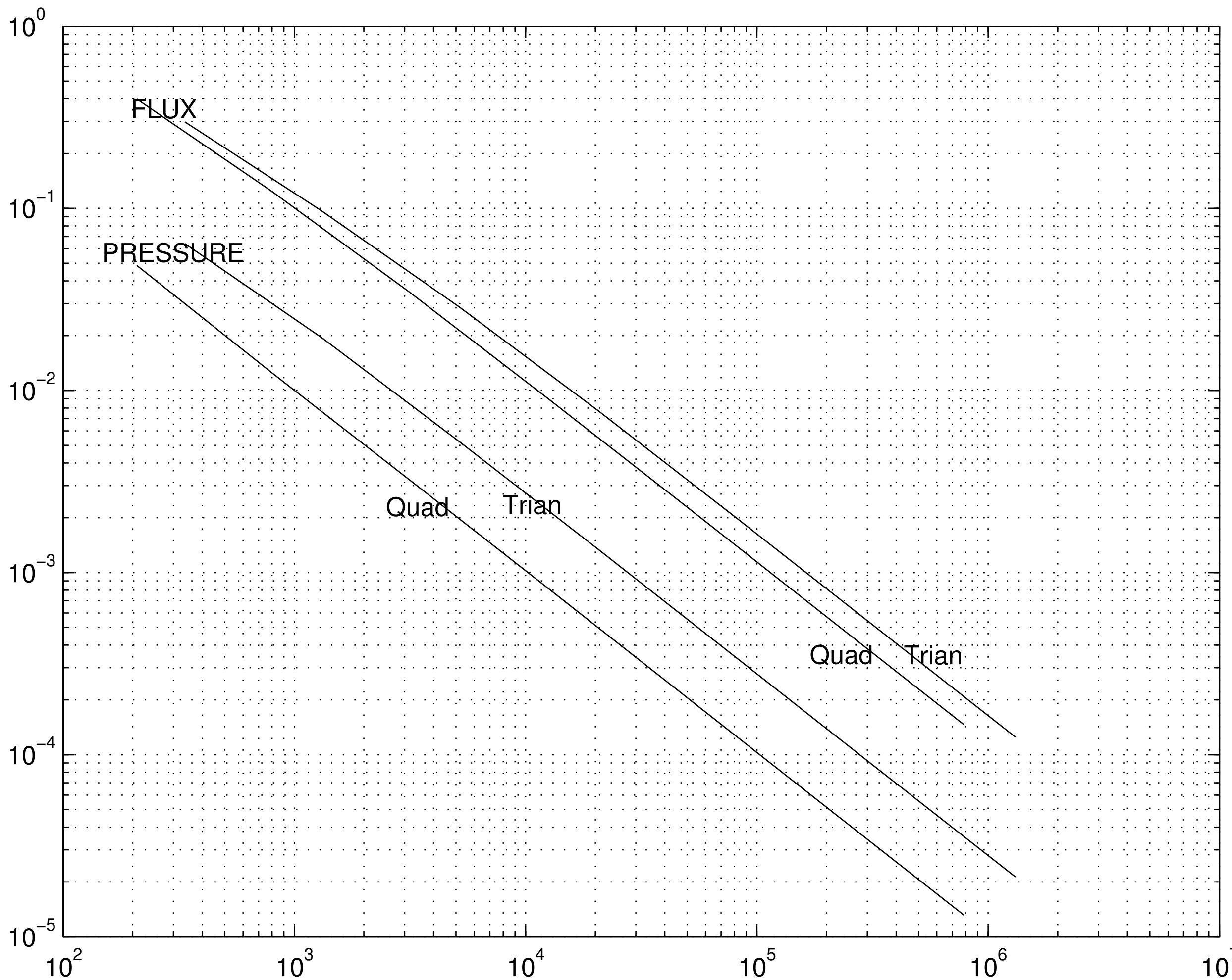
# Numerical experiments (3/4)





| $h^{-1}$ | modified RT FE | | SO FD | |
|---|---|---|---|---|
| | $\varepsilon_p$ | $\varepsilon_{\boldsymbol{f}}$ | $\varepsilon_p$ | $\varepsilon_{\boldsymbol{f}}$ |
| 16 | 1.58e-3 | 2.34e-2 | 1.61e-3 | 2.35e-2 |
| 32 | 7.95e-4 | 1.22e-2 | 7.99e-4 | 1.22e-2 |
| 64 | 3.98e-4 | 6.29e-3 | 3.99e-4 | 6.29e-3 |
| 128 | 1.99e-4 | 3.22e-3 | 1.99e-4 | 3.22e-3 |
| 256 | 9.97e-5 | 1.64e-3 | 9.97e-5 | 1.64e-3 |
| 512 | 4.98e-5 | 8.32e-4 | 4.98e-5 | 8.32e-4 |
| | $\varepsilon_p$ | $\varepsilon_{\boldsymbol{f}}$ | $\varepsilon_p$ | $\varepsilon_{\boldsymbol{f}}$ |
| 16 | 1.42e-3 | 2.24e-2 | 1.43e-3 | 2.25e-2 |
| 32 | 7.15e-4 | 1.17e-2 | 7.18e-4 | 1.17e-2 |
| 64 | 3.59e-4 | 5.96e-3 | 3.59e-4 | 5.98e-3 |
| 128 | 1.80e-4 | 3.06e-3 | 1.80e-4 | 3.07e-3 |
| 256 | 9.00e-5 | 1.56e-3 | 9.00e-5 | 1.56e-3 |
| 512 | 4.50e-5 | 7.93e-4 | 4.50e-5 | 7.93e-4 |

# Numerical experiments (4/4)




Accuracy of the mimetic discretizations versus the problem size.

# Conclusion

- the convergence rate of mimetic discretizations for the linear diffusion equation is optimal on both smooth and non-smooth meshes;
- asymptotically, the SO and FE methods result in the same discretization errors; however, the FE method requires a very accurate quadrature rule for integrating RT finite elements; the methods are identical if $[f^d, g^d]_X = (f^h, g^h)$.
- similar methodology can be used to obtaing superconvergence estimates;
- application of our methodology is limited to triangular and quadrilateral meshes.